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On-Line Timetable Rescheduling in a Transit Line

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Abstract. Public transportation systems in metropolitan areas carry a high density of daily traffic, heterogeneously distributed, and exposed to the negative consequences derived from service disruptions. Breakdowns, accidents, strikes, etc., require on-line operation adjustments to address these incidents and thus reduce their side effects, such as passenger extra-waiting times, complaints, potential operational dangers, etc. The Vehicle Rescheduling Problem consists of defining a new schedule for a set of previously scheduled trips, given that one/several trips cannot be carried out. This paper addresses the rescheduling problem in a transit line that has suffered a fleet size reduction (also denoted as Reduced Fleet Rescheduling Problem). We present different modeling possibilities depending on the assumptions that must be included in the modelization and we show that the problem can be rapidly solved using a reformulation that will be proven to have the integrality property. We test our results in a testbed of random instances outperforming previous results in the literature. We also include a real-world case of the commuter trains of Madrid, Spain to illustrate our solutions.

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1. Introduction

Timetable design is a central problem in transportation planning including many interfaces with other classical problems, e.g., line planning, vehicle scheduling, and vehicle rescheduling. The Transit Network Timetabling and Scheduling Problem (TNTSP) is devoted to obtaining and optimizing departure and arrival times for each line run (i.e., trip from an origin to a final destination) to and from each station over a planning horizon imposing/optimizing different constraints and objectives (Ibarra-Rojas, Giesen, and Rios-Solis 2014). The TNTSP is based on the following general input: An infrastructure of a transport system described by a node set (network stations) and an edge set (i.e., roads/tracks between adjacent stations), a trip demand matrix between pairs of nodes of the infrastructure, a set of transit lines with associated frequencies that have already been determined to satisfy such trip demand and, finally, a vehicle fleet with specific characteristics. The objective of the TNTSP is finding the arrival and departure times of each vehicle at each station such that the demand satisfaction, required fleet size, and vehicle capacities can be optimized.

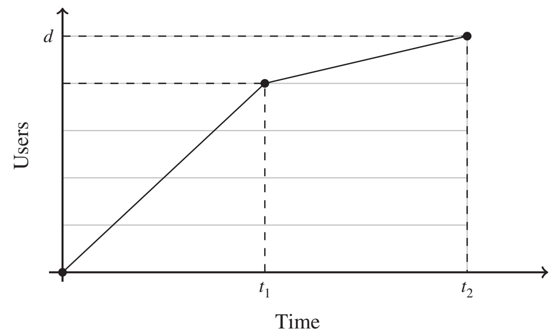
Accidents, strikes, and other sources of vehicle delays or cancellations may force modification of an initial timetable when vehicles in some sections cannot run according to the initial planning. Here disturbances are relatively small perturbations of the transit network that can be handled by modifying the timetable, but without modifying the duties of vehicles and drivers. Complementarily, disruptions are relatively large incidents requiring modification of the timetable and vehicle duties. There are several examples of possible disruptions that require the rescheduling of vehicles. Examples include: (1) interruptions coming from severe weather conditions, accidents, and the blockage of road or track sections (Cacchiani et al. 2012) or (2) fleet size reductions coming from vehicle breakdowns, drivers' and crew strikes (van Exel and Rietveld 2001) or vehicle reallocations made to reinforce other sections of the transit network (Burdett and Kozan 2009). A timetable may also become infeasible simply due to a heavy passenger flow (Mesa, Ortega, and Pozo 2009b).

To address these incidents, on-line operation adjustments are required to reduce their side effects. The Vehicle Rescheduling Problem (VRP) consists of defining a new timetable for a set of previously scheduled trips, given that one/several trips cannot be carried out. While many objectives and constraints remain from the timetabling problem, new requirements and objectives arise in this context. In terms of passenger transportation, the main decisions concern minimization of the deviations from the initial timetable in operation. This is done by cancelling some services and/or providing new reference times for some vehicles at specific points in the network (Spliet, Gabor, and Dekker 2014, Kroon, Maróti, and Nielsen 2015). Further decisions may concern reallocating other available resources. Li, Borenstein, and Mirchandani (2007) and Li, Mirchandani, and Borenstein (2009) define the Vehicle Rescheduling Problem (VRSP) as reassigning vehicles in real-time to a trip affected by a disabled vehicle as well as to other scheduled trips with given starting and ending times. In their model, they make binary decisions to cancel trips or to serve a trip with a vehicle coming from a depot (and therefore incurring a delay). In this paper, we address the VRP in a transit line that has suffered a fleet size reduction. Given an initial timetable with a fixed number of line runs in a directed transit line and a demand pattern, the problem consists of determining a new rescheduled timetable once the number of initial line runs has been reduced, and optimizing a demand inconvenience function. We also denote this problem as the Reduced Fleet Rescheduling Problem (RFRP).

Users plan their trips based on a known timetable, and can be greatly disturbed if the service does not arrive or depart at the expected time. When a disturbance occurs, such as a vehicle breakdown in a certain line, the system operator must make a decision about rescheduling the remaining vehicles that are normally operating along the network to reduce the loss of service quality perceived by the users. An important difference between the planning stage and the rescheduling stage during disruptions is that in the latter less time is available for making decisions. In principle, solutions are expected within minutes (on-line). For the resources, another important difference is that in general there is less flexibility in the rescheduling stage, since many resource duties have already started their scheduled operations when decisions of rescheduling must be assumed and cannot be easily diverted. In addition, the solution space is bounded by the remaining time until the end of the rescheduling horizon, which is usually the end of the day. Hence, if the disruption happens in the evening, then the solution space is much smaller than when the disruption happens in the morning. A straightforward *myopic* strategy consists of canceling those services that serve the least number of users. This methodology would not introduce any change/delay in the remaining timetables. Nevertheless, a recent paper by Mesa, Ortega, and Pozo (2013) has shown that if real time control strategies are applied along a transit corridor (i.e., by allowing delays in some services of the initial timetable), then the demand satisfaction after rescheduling can be significantly increased.

Example 1. The following situation describes a toy example of the RFRS for illustrative purposes. Let s be a station in a directed transit line: Its demand pattern of

Figure 1. Demand Pattern of Total Arrivals at Station s



total arrivals is represented in Figure 1. We assume that initially two vehicles are scheduled to depart at times t_1 and t_2; therefore, according to the figure a demand $0.8 \cdot d$ is served at time t_1 and a demand $0.2 \cdot d$ is served at time t_2. If the transport manager has to reschedule the service by establishing a single vehicle departure from station s, he may decide among three options for this isolated scenario. The options are: (1) to keep the service that departs at t_1 and cancel the one at t_2, (2) to keep the service that departs at t_2 and cancel the one at t_1 or (3) to delay the service that departs at t_1 within time interval (t_1, t_2) and cancel the service departing at t_2. The first and second options affect 20% and 80% of the demand, respectively. The third option represents a trade-off between the first two. Later we extend this example to reschedule a complete timetable in a transit line assuming a more complex demand pattern. □

In this paper, we address the RFRP to reschedule a timetable in a transit line. We describe a demand pattern to reflect passengers' behavior when some vehicle services are delayed or canceled. This pattern lets us derive a vehicle rescheduling framework coming from a timetabling formulation in a transit line. We present and compare different modeling approaches depending on the assumptions that must be included in the modelization and we show that the problem can be rapidly solved using a reformulation that will be proven to have the integrality property.

The remainder of the paper is organized as follows. Section 2 reviews the most relevant contributions related to this work. Section 3 presents the description of the problem and all details to compute the demand pattern in the rescheduling phase. Section 4 presents and compares different RFRP mathematical programming formulations. Computational experiments are provided in Section 5 to show the usefulness and applicability of this methodology. Finally, some conclusions are presented in Section 6.

2. Background

Different approaches have been developed in the literature to address the rescheduling problem distinguishing between (1) disturbances and disruptions, (2) the level of detail considered in the railway system, in particular in the timetable, and (3) focusing the objective on the vehicles or on the customers. In the second distinction two approaches can be distinguished, i.e., microscopic and macroscopic. The latter considers the transit network at a higher level in which stations can be represented by nodes of a graph and roads/tracks by arcs, and the details of block sections and signals are not taken into account. In a microscopic approach these aspects are considered in detail. In the case of railway systems, most of the approaches in the literature address (1) disturbances affecting the railway system rather than disruptions, (2) the railway system at a microscopic level rather than at a macroscopic level, and (3) minimizing the delays of trains or the number of canceled vehicles rather than minimizing the negative effects of disturbances and disruptions for passengers (see Cacchiani et al. 2014). This section focuses on disruptions at a macroscopic level (see, e.g., Veelenturf et al. 2016) in a transit line.

As to the timetabling problem in a transit line we refer to Mesa, Ortega, and Pozo (2014a) and the references therein. Gathering the integration of timetables, vehicle scheduling, and passenger choices, Mesa, Ortega, and Pozo (2014a) present a new approach for jointly planning timetables and vehicle schedules along a single transit line emphasizing the point of view of potential customers. A p-median based formulation is proposed for a given fleet size of vehicles. In addition, demand behavior is associated with the inclusion of closest assignment type constraints.

Control strategies such as short turns, deadheads, and/or express services can be implemented for the timetabling adjustment in a transit linear corridor. Mesa, Ortega, and Pozo (2009a) develop an effective plan for allocating fleet frequencies at stops along a line based on three objectives, i.e., minimizing passenger overload, maximizing passenger mobility, and minimizing passenger loss. Schedules for decongesting and recovering the line are determined by means of optimization models. The methodology proposed was applied to real data of the commuter train system of Madrid, Spain. Kumazawa, Hara, and Koseki (2010) seek to minimize the dissatisfaction experienced by the passengers due to disturbances. They propose a rescheduling algorithm that calculates a value for the amount of passenger dissatisfaction due to disturbances on the Japanese railway network. In addition to a conventional passenger flow analysis, the passenger overflow, defined as the waiting time experienced by a passenger while waiting on the platform, is considered. Nakamura, Hirai, and Nishioka (2011) present an algorithm for train rescheduling during disruptions which takes as input train groups, train cancellation sections, and return patterns. These factors are predetermined by the dispatchers. In case of a disruption obstructing a section of the network, the developed algorithm determines a new timetable by canceling trains, combining return patterns, and changing the train departure order at stations in a series of steps. The efficiency of the rescheduling plan is evaluated in terms of passenger dissatisfaction caused by propagated delays. The algorithm is tested on a railway line in a metropolitan area in Japan. Sato, Tamura, and Tomii (2013) presents a timetable rescheduling algorithm based on a mixed integer programming (MIP) formulation when train traffic is disrupted. They minimize further inconvenience to passengers instead of consecutive delays caused by the disruption, since loss of time and passenger satisfaction are implicitly and insufficiently considered in that paper. They presume that inconvenience of traveling by train consists of the traveling time on board, the waiting time at platforms, and the number of transfers. Hence, the objective function is calculated on the positive difference between the inconvenience that each passenger suffers along his route for a rescheduled timetable compared to a planned timetable. For instance, the inconvenience-minimized rescheduling is often achieved at the cost of further train delays. Some trains stay longer at a station to wait for extra passengers or to keep a connection.

Mesa, Ortega, and Pozo (2013) and Mesa et al. (2014b) study the RFRP reducing the current supply along one transportation line to reinforce the service of another line, exploited by the same public operator, which has suffered an incident or emergency. A methodology based on a geometric representation of solutions is presented. It allows the use of discrete optimization techniques to cover the underlying demand with efficiency criteria in this context of unexpected incidents. The proposed methodology is computationally tested and applied to real data of the commuter train system of Madrid.

This paper differs from all previously cited references in several aspects, which we believe provide significant contributions to the field. First, we describe the problem by providing a users' demand pattern for modeling the arrival pattern and the passengers' inconvenience function after rescheduling. This setting extends the one in Mesa, Ortega, and Pozo (2013) and Mesa et al. (2014b) to a general framework in a transit line. Second, a modeling framework for rescheduling the line is derived from a TNTSP formulation and we present and compare different modeling approaches depending on the assumptions that need to be included. We show that the problem can be rapidly solved by using a reformulation that will be proven to have the integrality property. We test our results in a testbed of random instances outperforming previous results in the literature. An experimental study, based on a line segment of the Madrid Regional

Railway network, shows that our proposed approach provides optimal reassignment decisions within computation times compatible with real-time use.

3. Problem Description

3.1. Transportation Supply: Infrastructure and Timetables

Let l be a directed transit line running along a set of stations $S = \{1, 2, \ldots, |S|\}$. We also consider an additional terminal station numbered as $|S|+1$. Each station $s \in S$ also occupies the position s along the transit line l. We denote by $\langle s \rangle$ the "name" of station s so that it could be a text string (e.g., $\langle 4 \rangle$ = "central station") or a number (e.g., $\langle 4 \rangle = 312$). Note that here l can also be understood as a general itinerary along a set of stations in a transit network. Each vehicle $k \in K$ (where $|K| = \kappa$) circulates along l during a time horizon that will be discretized into a set of time slots $T = \{1, \ldots, |T|\}$ performing a single line run or expedition along the line. Then, a timetable Θ is a set of arrival/departure times at each station for each vehicle, $\Theta = \{(\theta^+_{sk}, \theta^-_{sk}), s \in S, k \in K\}$. Denoting by λ_{sk} the waiting time of vehicle k at station s and by μ_{sk} the travel time from station s to station $s+1$, we define λ_{sk} and μ_{sk} as:

1. $\lambda_{sk} = \theta^-_{sk} - \theta^+_{sk}$, $s \in S, k \in K$.
2. $\mu_{sk} = \theta^+_{s+1,k} - \theta^-_{sk}$, $s \in S, k \in K$.

A timetable Θ defined by variables $\theta^+, \theta^-, \lambda$, and μ is called the *arrival-departure timetable*. Without loss of generality we can assume that $\theta^+_{1k} = \theta^-_{1k} - 1$, $\theta^+_{|S|+1k} = \theta^-_{|S|k} + \mu_{|S|k}$, and $\theta^+_{sk}, \theta^-_{sk} \in T$ for all $s \in S$, $k \in K$. In addition we denote by λ^* (λ_*) the maximum (minimum) waiting time that a vehicle can stay in a station, that is $\lambda_* \le \lambda_{sk} \le \lambda^*$.

Potentially, every timetable Θ can be generated over the sets S, K. Nevertheless, the number of feasible timetables can be greatly reduced under different considerations. First, if we assume equal speed for all vehicles and that vehicle overtakings are not allowed, a timetable can be redefined as $\Theta \equiv x = \{x_{st}, t \in T, s \in S\}$ where $x_{st} \in \{0, 1\}$ equals 1 if and only if a vehicle departs from station s at time t. A timetable Θ defined by variables x is called the *departure timetable* and it verifies:

1. $\theta^-_{sk} = \{t\colon x_{st} = 1, t \in T\}|_{(k)}$, where $\cdot|_{(k)}$ denotes the kth element of a set that is sorted in nondecreasing order.
2. $\theta^+_{sk} = \theta^-_{s-1k} + \mu_{s-1k}$.
3. $\lambda_{sk} = \theta^-_{sk} - \theta^+_{sk}$.

Second, the dimension of x can also be reduced defining a time window $T_s \subset T$ on each station $s \in S$. The elements in the set T_s are the time slots that are admissible to reach station s from station 1 and station $|S|$ from station s within the time horizon. To compute T_s for each $s \in S$ a minimum waiting time $\lambda_* = 1$ can be established at each station. In addition, we can define the time window of feasible time slots to arrive at and depart from each station. Let T^2_s be a time window that contains the set of time slot pairs that are feasible for arriving and departing at the station $s \in S$. In particular,

$$T^2_s = \begin{cases} \{(1, t)\colon t \in T_1\}, & \text{if } s = 1; \\ \{(u, t)\colon u - \mu_{s-1} \in T_{s-1},\, t \in T_s\colon \\ \quad u + \lambda_* \le t \le u + \lambda^*\} & \text{if } s \in S\colon s > 1. \end{cases} \tag{1}$$

Finally, using the previous assumption of equal speed for all vehicles, the time horizon T can be reduced without loss of generality (and therefore the dimension of x) and the travel times between stations transformed to a constant value. This transformation is achieved by redefining $\theta^-_{sk} := \theta^-_{sk} - \sum_{s' \in S\colon s' < s} \mu_{s'k}$, $s \in S\colon 1 < s$.

Example 2. Table 1 shows an arrival-departure timetable for two vehicles ($\kappa = 2$) in a directed transit line running along stations 27, 9, 15, and 11 that occupy positions 1, 2, 3, and 4 in the line, respectively. We also show the departure timetable (in terms of variables x) assuming no overtakings and equal speed for all vehicles. Given $|T| = 23$ we can compute the time windows T_s for each station as it is indicated.

3.2. Passengers' Demand

Passengers enter a station s and wait until a vehicle arrives. Let a_{st} be the number of passengers who access station s at time t. Demand a_{st} is served by the next vehicle that departs (strictly) after time $t-1$ (denoted by vehicle $k_{st} \in K$). Because vehicles are assumed to have unlimited capacity, once a vehicle leaves a station all passengers waiting at the station leave with it. We assume that passengers who entered a station at time t' suffer an inconvenience $\varphi_{st't} \in [0, 1]$ if they have to wait until time $t > t'$ for leaving/departing. Thus, each user suffers a normalized inconvenience in the

Table 1. Two Representations of a Timetable in a Directed Transit Line

		$k=1$			$k=2$				
s	$\langle s \rangle$	$(\theta^+_{sk}, \theta^-_{sk})$	λ_{sk}	μ_{sk}	$(\theta^+_{sk}, \theta^-_{sk})$	λ_{sk}	μ_{sk}	x	T_s
1	27	(1, 2)	1	2	(1, 8)	7	2	(x_{st}): $x_{st} = 1$, $t \in \{2, 8\}$; $x_{st} = 0$, $o.c.$	$\{2, \ldots, 9\}$
2	9	(4, 7)	3	4	(10, 12)	2	4	(x_{st}): $x_{st} = 1$, $t \in \{6, 11\}$; $x_{st} = 0$, $o.c.$	$\{4, \ldots, 13\}$
3	15	(11, 13)	2	3	(16, 19)	3	3	(x_{st}): $x_{st} = 1$, $t \in \{9, 15\}$; $x_{st} = 0$, $o.c.$	$\{6, \ldots, 15\}$
4	11	(16, 23)	7	0	(22, 23)	1	0	(x not defined for $s = \|S\|+1$)	

$[0,1]$ interval. Without loss of generality we can set the inconvenience to equal zero if t is not greater than a given threshold $\tau^1_{st'}$. This means that passengers may wait a certain amount of time $\tau^1_{st'}$ without suffering any inconvenience. On the other hand, we can assume that the inconvenience is maximum ($\varphi_{st't}=1$) after a time $t \ge \tau^2_{st'}$. Finally, within the interval $(\tau^1_{st'}, \tau^2_{st'})$ the inconvenience is assumed to be a nonnegative, nondecreasing monotone function of t, $\alpha_{st't} \in [0,1)$. In this way, φ fits the expression

$$\varphi_{st't} = \begin{cases} 0, & t' < t \le \tau^1_{st'}; \\ \alpha_{st't} \in [0,1), & \tau^1_{st'} < t < \tau^2_{st'}; \\ 1, & \tau^2_{st'} \le t. \end{cases} \tag{2}$$

Next, we model disruptions. In the event that a subset of vehicles becomes unavailable in K, the remaining set of vehicles $\bar{K} \subset K$ must be rescheduled and a new set of $\bar{\kappa} = |\bar{K}|$ departure times at each station must be determined (where $\bar{\kappa} = |\bar{K}| < \kappa = |K|$). In this situation, passengers ignore new departure times until they arrive at a station at time t. We denote by $k_{st'}$ the first vehicle, of the original timetable, with a departure from station s after t' and by $\theta^-_{sk_{st'}}$ such departure time. There are three possible decisions for each departure time $\theta^-_{s,k_{st'}}$ (or service) initially scheduled: (1) to keep the service in the initial timetable, (2) to delay the service within time interval $(\theta^-_{s,k_{st'}}, \theta^-_{s,k_{st'}+1})$ or (3) to cancel the service. In this way, the inconvenience $\tilde{\varphi}$ suffered by passengers arriving at time t' will be 0 if they depart from s as in normal operation, that is, t is no later than $\theta^-_{s,k_{st'}}$. Otherwise, if the departure from s is at a time t within time interval $(\theta^-_{s,k_{st'}}, \theta^-_{s,k_{st'}+1})$, passengers arriving at time t' will suffer an inconvenience denoted by the value $\tilde{\varphi}_{st't} = \alpha_{st't} \in [0,1)$. In addition, to penalize canceled services, we can assume that the inconvenience for passengers arriving at time t' is full (i.e., equal to 1) if the service that was initially scheduled at $\theta^-_{s,k_{st'}}$ is canceled. Therefore, the inconvenience function under disruptions, $\tilde{\varphi}$, is given by $\varphi_{st't}$ with $\tau^1_{st'} = \theta^-_{s,k_{st'}}$ and $\tau^2_{st'} = \theta^-_{s,k_{st'}+1}$

$$\tilde{\varphi}_{st't} = \begin{cases} 0, & t' < t \le \theta^-_{s,k_{st'}}; \\ \alpha_{st't} \in [0,1), & \theta^-_{s,k_{st'}} < t < \theta^-_{s,k_{st'}+1}; \\ 1, & \theta^-_{s,k_{st'}+1} \le t. \end{cases} \tag{3}$$

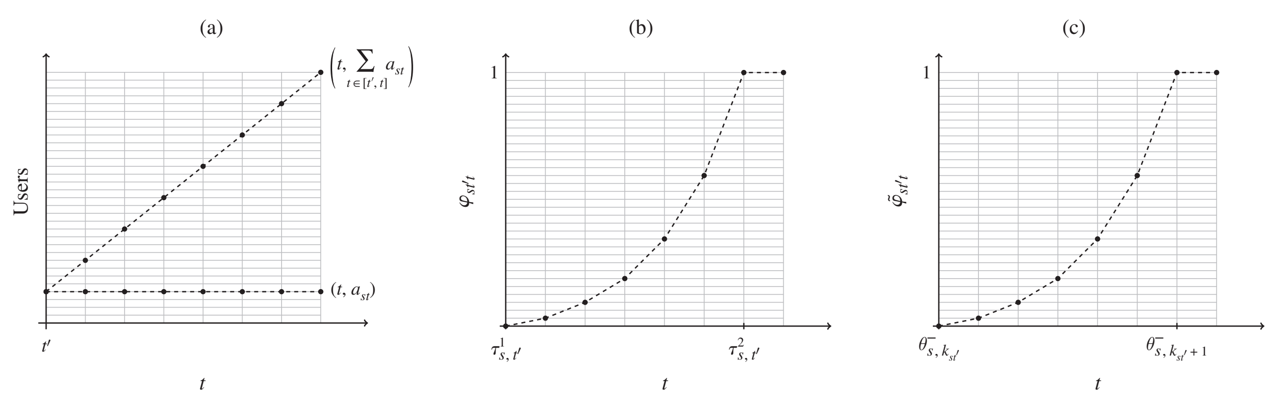
Example 3. Figure 2(a) shows an example of a constant arrivals pattern. The total number of users waiting at station s during time interval $[t', t]$ (that is $\sum_{t \in [t',t]} a_{st}$) is also depicted. Figure 2(b) shows an example of an inconvenience cost function for those users who arrived at time slot t' and suffered an inconvenience after time $\tau^1_{s,t'}$. Note that at time t, with $t' < t \le \tau^1_{s,t'}$, the inconvenience remains constant with value equal to 0, and at time t, with $t \ge \tau^2_{s,t'} = \tau^1_{s,t'} + 6$, the inconvenience remains constant (maximum) with value equal to 1. Figure 2(c) shows an example of an inconvenience cost function after a disruption. This is a particular case of inconvenience cost function with $\tau^1_{st'} = \theta^-_{s,k_{st'}}$ and $\tau^2_{st'} = \theta^-_{s,k_{st'}+1}$.

Note that in the RFRP, the only meaningful options are to keep, delay or cancel timetables for each line run. This implies that fleet sizes do not have to be increased, no additional drivers are required, etc., so the scenario under consideration (a single directed line) would not generate scheduling costs in terms of connections lost, transfer delays, etc. Therefore, the only costs involved in the RFRP come from the demand inconvenience function.

We conclude this section by summarizing the main assumptions and constraints that will later define the mathematical formulation of the problem. In particular,

- All vehicles run along a directed transit line (that can also be considered as an itinerary along a set of stations).
- Stopping time at stations and circulation speed is considered equal for all vehicles.
- Demand inconvenience is given by $\tilde{\varphi}$ and this is the only cost that will be optimized.
- An extra supply is not provided in the rescheduling, so no more than one vehicle can depart in the

Figure 2. Demand Patterns (a), Inconvenience Function (b), and Inconvenience Function After a Disruption (c)



interval $(\theta^-_{s,k_{st'}}, \theta^-_{s,k_{st'}+1})$ and if a vehicle departure is delayed, the next one must be canceled.

Therefore, no headway constraints are explicitly considered even when we impose that no more than one vehicle can depart in the interval $(\theta^-_{s,k_{st'}}, \theta^-_{s,k_{st'}+1})$.

4. Problem Formulation

In this section we present a catalogue of valid MIP formulations for the RFRP. We begin by developing a formulation for the TNTSP in a directed transit line, F^{xy}, that describes passenger inconvenience by means of function $\varphi_{st't}$. The F^{xy} gives rise to an RFRP formulation (F^{st}) if the inconvenience cost function under disruption, $\tilde{\varphi}_{st't}$, is considered instead of $\varphi_{st't}$. Next, we extend F^{st} into another formulation $\vec{F}^{sut}$ whose coefficient matrix we prove is totally unimodular. In addition, we show that the polyhedron of feasible solutions of $\vec{F}^{sut}$ can be projected onto a lower dimension space giving rise to an RFRP formulation (F^{sut}) with fewer variables and constraints. Finally, other extensions are considered for the case when vehicles are not identical.

4.1. A Timetabling Formulation for a Transit Line (F^{xy})

Our main goal here is to analyze the rescheduling problem that occurs whenever some event happens in a prespecified timetable. Therefore, one of our inputs must be an existing running timetabling. Before we address the main goal, however, we elaborate on the initial problem of determining an initial timetabling. Next, we present a new timetabling formulation that extends the one in Mesa, Ortega, and Pozo (2014a) by allowing variable stopping times of vehicles at intermediate stations. This results in a better adjustment of the demand. Once the arrival-departure timetable is determined by the considered formulation, it will be used as an input for the RFRP.

Next, we consider scheduling, timetabling, and vehicle scheduling as synonyms since each vehicle performs a single line-run along the transit line. Therefore κ vehicles are scheduled along the line and the demand arriving at station s at time t' is assigned to the first vehicle departing from that station after time t'. We recall that x_{st} is defined as a binary variable equal to 1 if a vehicle departs from station s at time t. The new formulation F^{xy} requires a binary variable that assigns passengers to a time slot where there must exist a vehicle departure. Then, let $y_{st't}$ be a binary variable equal to 1 if passengers arriving at station s at time t' are allocated to a vehicle departing at time t. The new timetabling formulation F^{xy} results in

$$F^{xy}: \quad \min \left\{ \sum_{s\in S} \sum_{t'\in T_s} \left(\sum_{t\in T_s:\, t'<t} a_{st'}\varphi_{st't} y_{st't} + a_{st'}\left(1 - \sum_{t\in T_s:\, t'<t} y_{st't}\right)\right)\right\} \tag{4a}$$

$$\text{s.t.} \quad \sum_{t\in T_1} x_{1t} = \kappa, \tag{4b}$$

$$0 \le \sum_{t'\in T_s: t'\le t} x_{st'} - \sum_{t'\in T_{s+1}: t'\le t+\mu_s} x_{s+1t'} \le 1 \qquad s\in S,\ t\in T_s: s<|S|, \tag{4c}$$

$$x_{st} \le \sum_{t': (t+\mu_s, t')\in T^2_{s+1}} x_{s+1t'} \quad s\in S: s<|S|,\ t\in T_s, \tag{4d}$$

$$y_{st't} \le x_{st} \quad s\in S,\ t',t\in T_s: t'\le t, \tag{4e}$$

$$\sum_{t\in T_s: t>t'} y_{st't} \le 1 \quad s\in S,\ t'\in T_s, \tag{4f}$$

$$x_{st}\in\{0,1\} \quad s\in S,\ t\in T_s, \tag{4g}$$

$$y_{st't}\in\{0,1\} \quad s\in S,\ t',t\in T_s: t'\le t. \tag{4h}$$

The objective function (4a) minimizes the total users' inconvenience. It indicates that the inconvenience of passengers who arrived at station s at time t' is $\varphi_{st't}$ if they are allocated to a vehicle departing at time t. Otherwise, if demand $a_{st'}$ is not allocated to any time slot, the inconvenience for those passengers will be full. Constraints (4b) impose that all vehicles depart from the first station (that is, there are κ departures and, in total, κ line runs along the line). Alternatively, (4b) can be strengthened imposing that all vehicles must depart from each station. Constraints (4c) impose that two vehicles cannot coincide in the same station and that the number of vehicles that have departed from a station cannot be higher than the number that have arrived. Constraints (4d) impose that if a vehicle departs from station s at time t, then another vehicle must depart from the next station inside time interval $[t+\mu_s+\lambda_*, t+\mu_s+\lambda^*]$ (recall that $\theta^-_{s,0}=0$, $\theta^-_{s,\kappa+1} = |T_s|+1$). Constraints (4e) ensure that no passenger allocations are made to timetables that do not exist. Constraints (4f) impose that each demand $a_{st'}$ is allocated to no more than one line run.

Formulation F^{xy} comes from the one developed in Mesa, Ortega, and Pozo (2014a) where the timetabling problem is seen as an extended p-median problem in an *ad-hoc* space. This new formulation, F^{xy}, extends the flexibility of the timetable since vehicles can remain stopped at intermediate stations allowing a better adjustment of the demand. In addition, since the objective function (4a) accounts for the global inconvenience of the scheduling, we can equivalently represent global satisfaction of the scheduling by subtracting this amount from the maximal possible inconvenience $A = \sum_{s\in S}\sum_{t'\in T} a_{st'}$. This way, we can also write F^{xy} as

$$F^{xy}: \quad A - \min \left\{ \sum_{s\in S} \sum_{t'\in T_s} \left(\sum_{t\in T_s: t'<t} a_{st'}\varphi_{st't} y_{st't} + a_{st'}\left(1 - \sum_{t\in T_s: t'<t} y_{st't}\right)\right)\right\} \tag{4a$'$}$$

s.t. (4b)–(4h).

Clearly, the objective (4a′) is equivalent to

$$\max \sum_{s\in S}\sum_{t'\in T_s}\sum_{t\in T_s:t'<t} a_{st'}(1-\varphi_{st't})y_{st't}. \qquad (4a'')$$

Indeed, the equivalence is straightforward from transforming (4a′) into (4a″)

$$\begin{aligned} A-\min&\Bigg\{\sum_{s\in S}\sum_{t'\in T_s}\Bigg(\sum_{t\in T_s:t'<t} a_{st'}\varphi_{st't}y_{st't} \\ &\qquad + a_{st'}\Bigg(1-\sum_{t\in T_s:t'<t} y_{st't}\Bigg)\Bigg)\Bigg\} \\ = A-\min&\sum_{s\in S}\sum_{t'\in T_s}\Bigg(a_{st'}+\sum_{t\in T_s:t'<t} a_{st'}(\varphi_{st't}-1)y_{st't}\Bigg) \\ = A-\min&\Bigg(A-\sum_{s\in S}\sum_{t'\in T_s}\sum_{t\in T_s:t'<t} a_{st'}(1-\varphi_{st't})y_{st't}\Bigg) \\ = \max&\sum_{s\in S}\sum_{t'\in T_s}\sum_{t\in T_s:t'<t} a_{st'}(1-\varphi_{st't})y_{st't}. \end{aligned}$$

4.2. RFRP Formulation (F^{st})

In this section, we derive a new valid RFRP formulation to reschedule a timetable (for example, a timetable generated with formulation F^{xy}) after some disruption. To compare results with those from previous literature, we (equivalently) change the objective function from a minimization of users' inconvenience to a maximization of users' satisfaction after the disruption, where we understand the satisfaction function (after disruption) as $1-\tilde{\varphi}$

$$F^{st}: \quad \max \sum_{s\in S}\sum_{t'\in T_s}\sum_{t\in T_s:t'<t<\theta^-_{s,k_{s,t'}+1}} a_{st'}(1-\tilde{\varphi}_{st't})x_{st} \qquad (6a)$$

$$\text{s.t.} \quad \sum_{t\in T_1} x_{1t}=\bar{\kappa}, \qquad (6b)$$

$$x_{st}\le \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} x_{s+1t'} \quad s\in S: s<|S|, t\in T_s, \qquad (6c)$$

$$\sum_{t\in T_s:\theta^-_{s,k-1}<t<\theta^-_{s,k+1}} x_{st}\le 1 \quad s\in S, k\in K, \qquad (6d)$$

$$0\le x_{st}\le 1 \quad s\in S, t\in T_s, \qquad (6e)$$

$$x_{st}\in\{0,1\} \quad s\in S, t\in T_s. \qquad (6f)$$

The objective function (6a) maximizes the total users' satisfaction. It indicates that the satisfaction of passengers who arrived at station s at time t' is $1-\varphi$ if a vehicle departs at time t. Note that outside the interval $(t', \theta^-_{s,k_{s,t'}+1})$ the satisfaction for demand $a_{st'}$ is not defined but results in being computed as 0. Note also that (6a) is well defined since constraints (6d) ensure that no more than one vehicle is rescheduled inside interval $(\theta^-_{s,k-1}, \theta^-_{s,k+1})$, thus avoiding demand $a_{st'}$ being served by more than one vehicle. This constraint also ensures that if a vehicle departure is delayed, the next one has to be canceled. As in (4b), constraints (6b) impose that all vehicles depart from the first station. As in (4c), constraints (6c) impose that if a vehicle departs from station s at time t, then another vehicle must depart from the next station inside time interval $[t+\mu_s+\lambda_*, t+\mu_s+\lambda^*]$.

Note that F^{st} generates a rescheduled timetable that is the same as the one given by $F^{xy}\cup\{(6d)\}$ when $\varphi=\tilde{\varphi}$. However, formulation F^{st} enjoys other modeling advantages and solution possibilities that we describe in Section 4.3. The main advantage is that by using $\tilde{\varphi}$ instead of φ, demand allocation variables y are no longer necessary since once a timetable is determined in F^{st} (in terms of x variables) the cost structure of $\tilde{\varphi}$ avoids having to allocate demand $a_{st'}$ to more than one vehicle departure. In terms of facility location theory, F^{xy} presents a location-allocation problem where a timetable x must be along the time horizon and demand must be allocated to vehicle departures by means of y variables. On the other hand, F^{st} can be seen as a covering problem (Church and ReVelle 1974) where each timetable x covers a certain demand, but no demand is covered more than once.

4.3. RFRP Extended Formulation ($\vec{F}^{sut}$)

In this section we analyze the feasible region of formulation F^{st}. To describe the convex hull of its lattice points, we embed the polyhedron of F^{st} into a space of higher dimension. For that reason, we introduce a new set of variables z that account for the time at which a vehicle arrives at and departs from each station. Therefore, we denote by z_{sut} the binary variable equal to 1 if pair (u,t) of the time window T^2_s is used by a vehicle; 0, otherwise. Note that variables x and z are related by means of

$$x_{st}=\sum_{t':(t+\mu_s,t')\in T^2_{s+1}} z_{s+1,t+\mu_s,t'} \quad s\in S: s<|S|, t\in T_s, \qquad (7)$$

$$x_{st}=\sum_{u:(u,t)\in T^2_s} z_{sut}. \quad s\in S: 1<s, t\in T_s. \qquad (8)$$

Indeed, constraints (7) relate departure times of variables x with arrival times of variables z imposing that if a vehicle departs from station s at time t, then it arrives at station $s+1$ at time $t+\mu_s$ to depart at a time t' within the time window $(t+\mu_s, t')\in T^2_{s+1}$. Constraints (8) relate departure times of variables x with departures times of variables z imposing that if a vehicle departs from station s at time t (according to variables x), then it also departs at time t (according to variables z) having arrived at a time u within time window $(u,t)\in T^2_s$.

Conversely, due to the binary nature of variables z and x, they can be related by means of

$$z_{sut}=x_{s-1,u-\mu_{s-1}}x_{st} \quad s\in S: 1<s, (u,t)\in T^2_s. \qquad (9)$$

Hence, formulation F^{st} can be written by using variables z_{sut} as follows:

$$\vec{F}^{sut}: \quad \max \sum_{s\in S}\sum_{t'\in T_s}\sum_{t\in T_s:t'<t<\theta^-_{s,k_{s,t'}+1}} a_{st'}(1-\tilde{\varphi}_{st't})x_{st} \qquad (10a)$$

$$\text{s.t.} \quad \sum_{t\in T_1} x_{1t} = \bar{\kappa}, \tag{10b}$$

$$x_{st} = \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} z_{s+1,t+\mu_s,t'} \quad s\in S: s<|S|,\ t\in T_s, \tag{10c}$$

$$x_{st} = \sum_{u:(u,t)\in T^2_s} z_{sut} \quad s\in S: 1<s,\ t\in T_s, \tag{10d}$$

$$\sum_{t\in T_s:\theta^-_{s,k-1}<t<\theta^-_{s,k+1}} x_{st} \le 1 \quad s\in S,\ k\in K, \tag{10e}$$

$$0\le x_{st}\le 1 \quad s\in S,\ t\in T_s, \tag{10f}$$

$$0\le z_{sut}\le 1 \quad s\in S: 1<s,\ (u,t)\in T^2_s, \tag{10g}$$

$$x_{st}\in\{0,1\} \quad s\in S,\ t\in T_s, \tag{10h}$$

$$z_{sut}\in\{0,1\} \quad s\in S: 1<s,\ (u,t)\in T^2_s. \tag{10i}$$

Problem (10a)–(10i) can be seen as a constrained maximum cost flow problem in a directed network where $\bar{\kappa}$ units of flow (vehicles) are sent from station 1 to station $|S|$. Arcs can be considered trips between adjacent stations at a time instant and the cost of each edge is the captured demand when a vehicle departs at a certain time. Formulation $\vec{F}^{sut}$ differs from F^{st} in constraints (10c)–(10d) that preserve vehicle flows at each station in $\vec{F}^{sut}$ as (6c) do in F^{st}. Thus, the solution space in terms of variables x_{st} can be proven to be the same for $\vec{F}^{sut}$ and F^{st}.

Property 1. *Let Ω^{st} be the lattice points defined by constraints (6b)–(6f) and $\vec{\Omega}^{sut;x}$ the projection defined by constraints (10b)–(10i) over the x variables. Then $\Omega^{st} = \vec{\Omega}^{sut;x}$.*

Next, we extend the analysis to the polyhedra of the linear relaxations of both formulations.

Property 2. *Let Ω^{st}_{LR} be the polyhedron defined by constraints (6b)–(6e) and $\vec{\Omega}^{sut;x}_{LR}$ the projection defined by constraints (10b)–(10g) over the x variables. Then $\Omega^{st}_{LR} \supsetneq \vec{\Omega}^{sut;x}_{LR}$.*

The importance of the equivalent formulation $\vec{\Omega}^{sut}$ comes from our next result which proves that its constraints matrix is totally unimodular (TU) and therefore this problem is solvable with linear programming.

Property 3. *The RFRP can be solved with linear programming by means of $\vec{F}^{sut}$.*

Based on the last formulation $\vec{A}^{sut}$ one can derive another formulation that also enjoys the integrality property but with a smaller number of variables and constraints. Consider the following formulation:

$$F^{sut}: \quad \max\Bigg\{\sum_{t'\in T_1}\sum_{t'':(t+\mu_1,t'')\in T^2_2} a_{1t'}(1-\tilde{\varphi}_{1t't})z_{2,t+\mu_1,t''} + \sum_{s\in S}\sum_{t'\in T_s}\sum_{(u,t)\in T^2_s:t'<t<\theta^-_{s,k_{s,t'}+1}} a_{st'}(1-\tilde{\varphi}_{st't})z_{sut}\Bigg\} \tag{11a}$$

$$\text{s.t.} \quad \sum_{(u,t)\in T^2_2} z_{2ut} = \bar{\kappa}, \tag{11b}$$

$$\sum_{u:(u,t)\in T^2_s} z_{sut} = \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} z_{s+1,t+\mu_s,t'} \quad s\in S: 1<s<|S|,\ t\in T_s, \tag{11c}$$

$$\sum_{(u,t)\in T^2_s:\theta^-_{s,k-1}<t<\theta^-_{s,k+1}} z_{sut}\le 1 \quad s\in S: 1<s,\ k\in K, \tag{11d}$$

$$0\le z_{sut}\le 1 \quad s\in S: 1<s,\ (u,t)\in T^2_s, \tag{11e}$$

$$z_{sut}\in\{0,1\} \quad s\in S: 1<s,\ (u,t)\in T^2_s.. \tag{11f}$$

The objective function (11a) maximizes the total users' satisfaction. Since variables z are not defined for $s=1$, we count the departure time t from station 1 by means of $z_{2,t+\mu_s,t''}$ for a t'' within time window $(t+\mu_1, t'') \in T^2_2$. Constraints (11b) impose that on each station all vehicles are scheduled. Constraints (11c) impose that the number of vehicles departing from station s before time $t\in T_s$, must be lower than or equal to the number of vehicles departing from station $s+1$ at time $t+\mu_s$. Constraints (11d) impose that there is no more than one vehicle departing from station s inside time interval $(\theta^-_{s,k-1}, \theta^-_{s,k+1})$.

Property 4. *Let Ω^{sut} be the lattice points defined by constraints (11b)–(11f) and $\vec{\Omega}^{sut;z}$ the projection of (10b)–(10i) over the z variables. Then $\Omega^{sut} = \vec{\Omega}^{sut;z}$.*

Next, we extend the analysis to the polyhedra of the linear relaxations of both formulations.

Property 5. *Let Ω^{sut}_{LR} be the polyhedron defined by constraints (11b)–(11e) and $\vec{\Omega}^{sut;z}_{LR}$ the projection of (10b)–(10g) over the z variables. Then $\Omega^{sut}_{LR} = \vec{\Omega}^{sut;z}_{LR}$.*

Properties 4 and 5 imply that the coefficient matrix of the constraints defining Ω^{sut} is also TU as it is the corresponding matrix of $\vec{\Omega}^{sut}$. However, using formulation F^{sut} for solving problem RFRP is more convenient since it has fewer variables and constraints. Thus, we shall use it in our computational experiments.

4.4. Modeling Extensions

The models in the previous sections assume that all vehicles are identical. This assumption is reasonable but could be too restrictive in practical situations, specifically if further side constraints were added in terms of capacities and travel times. It is beyond the scope of this paper to discuss all those possible variations but it might be of interest to include the extensions of formulations F^{st} and F^{sut} to the case when we make distinctions among the different vehicles.

Let x_{stk} be a binary variable equal to 1 if vehicle $k\in\bar{K}$ (recall that $\bar{K}$ is the set of vehicles that are about to be rescheduled) departs from station s at time t (0 otherwise), and analogously z_{sutk} be a binary variable equal to 1 if vehicle k arrives at station s at time u and departs from s at time t. Note that index k may refer to vehicles and trips (or line runs) because every line run is carried out by a single vehicle and every vehicle carries out a single line run. We denote by F^{stk} (F^{sutk}) the formulation that can be derived straightforward from F^{st} (F^{sut}) considering variables x_{stk} (z_{sutk}) instead of x_{st} (z_{sut})

$$F^{stk}: \quad \max \sum_{k\in\bar{K}} \sum_{s\in S} \sum_{t'\in T_s} \sum_{t\in T_s: t'<t<\theta^-_{s,k_{s,t'}+1}} a_{st'}(1-\tilde{\varphi}_{st't})x_{stk} \tag{12a}$$

$$\text{s.t.} \quad \sum_{t\in T_1} x_{1tk} = 1 \quad k\in\bar{K}, \tag{12b}$$

$$x_{stk} \le \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} x_{s+1,t'k} \quad s\in S: s<|S|,\ t\in T_s, k\in\bar{K}, \tag{12c}$$

$$\sum_{k'\in K} \sum_{t\in T_s: \theta^-_{s,k-1}<t<\theta^-_{s,k+1}} x_{stk'} \le 1 \quad s\in S, k\in K, \tag{12d}$$

$$x_{stk}\in\{0,1\} \quad s\in S, t\in T_s, k\in\bar{K}, \tag{12e}$$

$$F^{sutk}: \quad \max\Bigg\{ \sum_{k\in\bar{K}} \sum_{t'\in T_1} \sum_{t'':(t+\mu_1,t'')\in T^2_2} a_{1t'}(1-\tilde{\varphi}_{1t't}) z_{2,t+\mu_1,t'',k} + \sum_{k\in\bar{K}} \sum_{s\in S} \sum_{t'\in T_s} \sum_{(u,t)\in T^2_s: t'<t<\theta^-_{s,k_{s,t'}+1}} a_{st'} \cdot (1-\tilde{\varphi}_{st't}) z_{sutk} \Bigg\} \tag{13a}$$

$$\text{s.t.} \quad \sum_{(u,t)\in T^2_1} z_{1utk} = 1 \quad k\in\bar{K}, \tag{13b}$$

$$\sum_{u:(u,t)\in T^2_s} z_{sutk} = \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} z_{s+1,t+\mu_s,t'k} \quad s\in S: s<|S|, t\in T_s, k\in\bar{K}, \tag{13c}$$

$$\sum_{k'\in\bar{K}} \sum_{(u,t)\in T^2_s: \theta^-_{s,k-1}<t<\theta^-_{s,k+1}} z_{sutk} \le 1 \quad s\in S, k\in K, \tag{13d}$$

$$z_{sutk}\in\{0,1\} \quad s\in S, (u,t)\in T^2_s, k\in\bar{K}. \tag{13e}$$

Note that the interpretation of constraints (12a)–(12e) and (13a)–(13e) is the same as those associated in formulations F^{st} and F^{sut}, respectively.

A formulation using index k that would own the integrality property would also admit additional constraints without losing such property (which is really important to define other modeling extensions). First, we could create/modify constraints by means of parameters depending on the index k, for example, by using the waiting time of vehicle k at station s (λ_{sk}) or the travel time of vehicle k from station s to station $s+1$ (μ_{sk}). Second, additional constraints could be added related to bounds on the variables, e.g., time windows on departing/arriving vehicles.

If no additional parameter/constraint dependent on index k is added to the model, F^{stk} contains a large set of symmetric optimal solutions ($\bar{\kappa}!$) since vehicles in an optimal solution can be relabeled without changing the objective value. In this way, a solution of F^{st} is related with $\kappa!$ solutions of F^{stk}. Therefore, F^{st} and F^{stk} contain the same set of non-symmetric solutions.

Mesa, Ortega, and Pozo (2013) provide an integer linear programming formulation valid for the RFRP, considering the same assumptions for the problem. While the objective function means the same as that presented in this paper, the cost vector used in Mesa, Ortega, and Pozo (2013) is now split into two components, i.e., the population that arrives at station s at time t' (given by parameter $a_{st'}$) and the convenience function (given by $1-\tilde{\varphi}_{st't}$). In this way, the demand modelization is more general and no logit function (see Mesa, Ortega, and Pozo 2013) must be defined. Therefore, formulation F^{sutk} contains the same set of solutions as those presented in Mesa, Ortega, and Pozo (2013) but the alternative flow conservation constraints (13c) provide a better performance and a significant improvement in running times as we show in Section 5.

We conclude this section by noting that the time discretization assumed in this paper is not an actual constraint. The reasons supporting our claim are threefold. First, no actual real-world timetable presents arrival-departure times in fractions of minutes. Second, specialized literature in previous contributions considers a discretized time horizon to model discrete arrivals of groups of passengers and departures/arrivals of vehicles (see Mesa, Ortega, and Pozo 2009b, 2013, 2014a; Mesa et al. 2014b). Third, a time-continuous approach would be easily approximated up to any degree of accuracy with the developments of this paper by simply introducing a sufficiently large number of time slots. This scenario would also be tractable with the development of the present paper showing how a rescheduling can be rapidly carried out in large instances, outperforming previous results in the literature and using real-world data.

5. Computational Experiments

5.1. Testbed of Random Instances

In this section, we assess the computational performance of the different models presented. We have generated instances similar to those in Mesa, Ortega, and Pozo (2013) to later establish a comparison with the RFRP formulations presented in that paper. Along a one-way transit line with a number $|S| = 10$ of stations we have generated random instances for $|I| =$ 1,000 transportation requests (origin-destination (O-D) trips) in the time horizons $|T| \in \{60, 120, 180, 240\}$ with desired arrival times following a uniform distribution. This time-dependent O-D matrix gives us the a_{st} values. The inconvenience cost function after disruption has been defined as follows:

$$\tilde{\varphi}_{st't} = \begin{cases} 0, & t'<t\le\theta^-_{s,k_{st'}}; \\ \left(\dfrac{t-\theta^-_{s,k_{st'}}}{\theta^-_{s,k_{st'}+1}-\theta^-_{s,k_{st'}}}\right)^p, & \theta^-_{s,k_{st'}}<t\le\theta^-_{s,k_{st'}+1}; \\ 1, & \theta^-_{s,k_{st'}+1}\le t. \end{cases} \tag{14}$$

This results in a discretized function as in Figure 2(c), assuming $p = 2$. This inconvenience function is similar to those described in Mesa, Ortega, and Pozo (2013, 2014a).

Each of our tables reports the following items: Each row corresponds to a group of five instances with the same characteristics ($|T|, \kappa, \bar{\kappa}$) indicated in the first three columns (recall that $|T|$ is the number of time slots in the time horizon, κ is the fleet size before the rescheduling, and $\bar{\kappa}$ is the fleet size to be rescheduled). Column $t/gap(\#)$ first reports the average running time in seconds of the five instances of the row. If none of the five instances were solved to optimality, this column reports the average relative gap (indicated by a percentage (%)) computed with the best solution found by the solver and the LP bound. In addition, if at least one instance reaches the CPU time limit, we indicate in parentheses the number of instances that could be solved to optimality within the time limit and, in these cases, we compute the average running time using the time limit for those instances that could not be solved to optimality. Column t^*/gap^* reports the biggest CPU time over the five instances of the group. Whenever the time limit is reached for at least one instance, the maximum relative gap (indicated by a percentage (%)) is reported instead. Column *gapLR* reports the average relative percentage gap (of the five instances of the row) computed with the best solution found by the solver and the optimal value of the linear relaxation at the root node. Column *nodes* indicates the average number of

Table 2. Computational Results Comparing the Rescheduling Formulations F^{st} and F^{sut}

			F^{st}					F^{sut}				
$\|T\|$	κ	$\bar{\kappa}$	$t/gap(\#)$	t^*/gap^*	*gapLR*	*nod*	*obj*	$t/gap(\#)$	t^*/gap^*	*gapLR*	*nod*	*obj*
60	4	1	0.2	0.7	10.77	**1**	**415.6**	**0.1**	**0.1**	**0**	**1**	**415.6**
60	4	2	**0**	**0.1**	1.21	**1**	**761.2**	0.1	**0.1**	**0**	**1**	**761.2**
60	4	3	0.2	0.5	1.56	**1**	**898**	**0.1**	**0.1**	**0**	**1**	**898**
60	4	4	**0**	**0**	**0**	**1**	**1,000**	0.1	0.1	**0**	**1**	**1,000**
120	9	1	1.3	3	11.34	**1**	**199.6**	**0.3**	**0.4**	**0**	**1**	**199.6**
120	9	2	7.1	9.2	10.69	1,081	**378.8**	**0.3**	**0.3**	**0**	**1**	**378.8**
120	9	3	10.9	18.9	8.6	1,167	**541.2**	**0.3**	**0.3**	**0**	**1**	**541.2**
120	9	4	5.8	12.3	3.99	101	**690.8**	**0.3**	**0.3**	**0**	**1**	**690.8**
120	9	5	2.2	8.2	1.76	7	**794.2**	**0.3**	**0.3**	**0**	**1**	**794.2**
120	9	6	0.9	3	1.43	3	**856.6**	**0.3**	**0.3**	**0**	**1**	**856.6**
120	9	7	0.4	0.5	0.99	**1**	**908.4**	**0.3**	**0.3**	**0**	**1**	**908.4**
120	9	8	0.4	1	0.47	**1**	**956.4**	**0.3**	**0.3**	**0**	**1**	**956.4**
120	9	9	**0.1**	**0.1**	**0**	**1**	**1,000**	0.3	0.3	**0**	**1**	**1,000**
180	14	1	6	12.2	15.02	6	**128.8**	**0.5**	**0.5**	**0**	**1**	**128.8**
180	14	2	19.2	37.6	13.39	4,076	**251.8**	**0.4**	**0.5**	**0**	**1**	**251.8**
180	14	3	151.5	489.7	11.7	33,718	**365.6**	**0.4**	**0.5**	**0**	**1**	**365.6**
180	14	4	633	1,384.2	10.4	220,693	**471.2**	**0.5**	**0.5**	**0**	**1**	**471.2**
180	14	5	548.4	1,692.9	8.95	74,771	**569.8**	**0.5**	**0.5**	**0**	**1**	**569.8**
180	14	6	84.1	245.5	6	9,289	**663.6**	**0.5**	**0.5**	**0**	**1**	**663.6**
180	14	7	15.2	33.5	3.21	288	**742.6**	**0.5**	**0.5**	**0**	**1**	**742.6**
180	14	8	4.1	8.4	1.94	6	**798.6**	**0.5**	**0.5**	**0**	**1**	**798.6**
180	14	9	2.2	6.6	1.42	2	**842.6**	**0.5**	**0.5**	**0**	**1**	**842.6**
180	14	10	1.4	2.4	1.26	**1**	**879.4**	**0.5**	**0.5**	**0**	**1**	**879.4**
180	14	11	1.5	2.9	1.15	**1**	**911.8**	**0.5**	**0.5**	**0**	**1**	**911.8**
180	14	12	1.3	2.7	0.78	**1**	**943.4**	**0.5**	**0.5**	**0**	**1**	**943.4**
180	14	13	0.6	0.9	0.41	**1**	**972.6**	**0.5**	**0.5**	**0**	**1**	**972.6**
180	14	14	**0.2**	**0.2**	**0**	**1**	**1,000**	0.4	0.4	**0**	**1**	**1,000**
240	18	1	10.3	13.6	16.31	18	**100**	**0.7**	**0.8**	**0**	**1**	**100**
240	18	2	63.4	143.6	16.06	10,678	**196**	**0.6**	**0.6**	**0**	**1**	**196**
240	18	3	579.2 (4)	4.95%	15.52	159,843	**286.2**	**0.6**	**0.6**	**0**	**1**	**286.2**
240	18	4	1,434.3 (2)	7.4%	14.41	180,966	371.4	**0.6**	**0.6**	**0**	**1**	**372.8**
240	18	5	1,563.2 (1)	5.75%	12.23	102,120	457.2	**0.6**	**0.6**	**0**	**1**	**457.6**
240	18	6	2.86% (0)	4.48%	10.76	81,552	535.4	**0.6**	**0.7**	**0**	**1**	**536**
240	18	7	1,166.1 (3)	2.4%	8	62,051	**612.4**	**0.6**	**0.7**	**0**	**1**	**612.4**
240	18	8	278.2	912	5.15	22,099	**683**	**0.6**	**0.6**	**0**	**1**	**683**
240	18	9	28	62.7	3.1	1,499	**740.6**	**0.6**	**0.7**	**0**	**1**	**740.6**
240	18	10	14	32.9	2.21	201	**783.4**	**0.6**	**0.7**	**0**	**1**	**783.4**
240	18	11	6.1	11.9	1.76	13	**819.6**	**0.6**	**0.7**	**0**	**1**	**819.6**
240	18	12	4.9	9.7	1.41	5	**852**	**0.6**	**0.7**	**0**	**1**	**852**
240	18	13	5	9	1.29	10	**880.2**	**0.6**	**0.7**	**0**	**1**	**880.2**
240	18	14	3.4	7	1.1	2	**906.8**	**0.7**	**0.7**	**0**	**1**	**906.8**
240	18	15	2.5	4.6	0.85	**1**	**932.2**	**0.6**	**0.7**	**0**	**1**	**932.2**
240	18	16	1.8	3.4	0.45	**1**	**957.2**	**0.6**	**0.6**	**0**	**1**	**957.2**
240	18	17	0.9	2.2	0.2	**1**	**979.8**	**0.6**	**0.6**	**0**	**1**	**979.8**
240	18	18	**0.3**	**0.3**	**0**	**1**	**1,000**	0.6	0.6	**0**	**1**	**1,000**

nodes explored in the branch and bound tree. Finally, column *obj* reports the average objective value of the five instances of the row. All tables report analogous items for the different formulations described throughout the paper. To facilitate the comparison among all tables, we have denoted the best result among all in the same group in bold.

All instances were solved with the MIP Xpress 7.7 optimizer, in a Windows 7 environment with an Intel Core i7 CPU 2.93 GHz processor and 16 GB RAM. Default values were initially used for all parameters of Xpress solver and a time limit of 3,600 seconds was set.

Table 2 reports the comparison between formulations F^{st} and F^{sut}. The caption just above each block gives the formulation to which the block refers. Even when F^{st} provides optimal solutions in small running times until $|T| = 120$, longer times are required for some instances of $|T| = 180$ and not all instances can be solved to optimality for some instances of $|T| = 240$. Column *gapLR* in F^{st} shows that only instances where $\bar{\kappa} = \kappa$ can

Table 3. Computational Results Comparing The 4-Index Rescheduling Formulation F^{sutk} as Presented in Mesa, Ortega, and Pozo (2013) with the Current Models F^{sut} and F^{sutk}

			F^{sutk} as in Mesa, Ortega, and Pozo (2013)					F^{stk}					F^{sutk}				
$\|T\|$	κ	$\bar{\kappa}$	*t/gap(#)*	*t*/gap**	*gapLR*	*nod*	*obj*	*t/gap(#)*	*t*/gap**	*gapLR*	*nod*	*obj*	*t/gap(#)*	*t*/gap**	*gapLR*	*nod*	*obj*
60	4	1	3	8.6	14.55	4	**415.6**	0.2	0.7	10.77	**1**	**415.6**	**0.1**	**0.1**	**0**	**1**	**415.6**
60	4	2	6.5	26.6	3.84	12	**761.2**	**0.3**	1.2	1.21	**1**	**761.2**	0.4	**0.4**	**0**	**1**	**761.2**
60	4	3	32	77.1	3.53	522	**898**	3.9	7.6	1.56	11	**898**	**0.5**	**0.6**	**0**	**1**	**898**
60	4	4	1.1	1.2	**0**	**1**	**1,000**	**0.3**	**0.3**	**0**	**1**	**1,000**	0.6	0.6	**0**	**1**	**1,000**
120	9	1	4.2	8.9	15.54	**1**	**199.6**	1.3	3.1	11.34	**1**	**199.6**	**0.6**	**0.6**	**0**	**1**	**199.6**
120	9	2	610	1,103.6	14.7	21,456	**378.8**	22.5	28.4	10.69	175	**378.8**	**0.9**	**0.9**	**0**	**1**	**378.8**
120	9	3	4% (0)	6.09%	12.36	43,180	**541.2**	81.7	123.2	8.6	4,138	**541.2**	**1.3**	**1.3**	**0**	**1**	**541.2**
120	9	4	1,765.9 (1)	5.78%	7.46	16,730	689.6	130.8	225.4	3.99	4,752	**690.8**	**1.8**	**1.9**	**0**	**1**	**690.8**
120	9	5	2.18% (0)	5.66%	5	8,236	793	420.3 (4)	0.75%	1.76	11,179	**794.2**	**2.3**	**3**	**0**	**1**	**794.2**
120	9	6	2.39% (0)	3.95%	4.75	10,199	854.8	475.3 (4)	0.67%	1.43	6,276	**856.6**	**2.9**	**3.6**	**0**	**1**	**856.6**
120	9	7	2.3% (0)	2.87%	3.77	8,179	908.2	504	1,072.6	0.99	5,345	**908.4**	**3.6**	**4.1**	**0**	**1**	**908.4**
120	9	8	1,462.1 (1)	1.47%	2.25	6,348	**956.4**	66.1	116.6	0.47	125	**956.4**	**4.2**	**5.2**	**0**	**1**	**956.4**
120	9	9	10.1	12.6	**0**	**1**	**1,000**	**1.4**	**1.5**	**0**	**1**	**1,000**	3.4	3.5	**0**	**1**	**1,000**
180	14	1	24.9	38	19.93	162	**128.8**	5.9	12	15.02	6	**128.8**	**1.2**	**1.2**	**0**	**1**	**128.8**
180	14	2	1,622.7 (1)	8.21%	18.01	34,985	**251.8**	50.5	58.6	13.39	1,778	**251.8**	**1.3**	**1.4**	**0**	**1**	**251.8**
180	14	3	8.91% (0)	11.4%	16.48	26,553	364.6	627.7 (4)	2.05%	11.7	33,292	**365.6**	**2.1**	**2.4**	**0**	**1**	**365.6**
180	14	4	9.46% (0)	11.83%	14.95	9,648	470.2	1,753.1 (1)	5.61%	10.4	57,412	**471.2**	**3.3**	**3.8**	**0**	**1**	**471.2**
180	14	5	8.91% (0)	11.4%	13.67	6,144	566.4	5.11% (0)	6.65%	8.95	51,530	**569.8**	**4.6**	**5.1**	**0**	**1**	**569.8**
180	14	6	7.91% (0)	10.61%	11.11	4,090	655.2	3.78% (0)	5.61%	6.03	18,722	**663.4**	**5.8**	**6.3**	**0**	**1**	**663.6**
180	14	7	7.39% (0)	8.6%	9.81	3,260	721.6	1.96% (0)	3.43%	3.27	6,701	**742.2**	**7.5**	**10.9**	**0**	**1**	**742.6**
180	14	8	9.3% (0)	16.83%	12.02	1,476	754.2	1,424.4 (2)	2.15%	1.94	3,405	**798.6**	**8.3**	**10.1**	**0**	**1**	**798.6**
180	14	9	6.14% (0)	9.33%	7.57	950	822.2	1,745.7 (1)	1.27%	1.44	3,294	**842.4**	**10.1**	**12**	**0**	**1**	**842.6**
180	14	10	5.48% (0)	6.79%	6.76	682	861.2	1,461.4 (1)	1.42%	1.38	1,822	**878.4**	**12.3**	**15**	**0**	**1**	**879.4**
180	14	11	3.96% (0)	4.72%	5.03	1,051	902.6	1,460.1 (1)	1.29%	1.17	1,696	**911.6**	**17.7**	**21.3**	**0**	**1**	**911.8**
180	14	12	3.26% (0)	5.03%	4.09	1,188	933.4	1,441.7 (1)	0.99%	0.81	2,100	**943.2**	**19**	**24**	**0**	**1**	**943.4**
180	14	13	1.27% (0)	1.68%	1.92	1,225	971.4	1,161.5 (2)	0.32%	0.41	5,263	**972.6**	**19**	**21.4**	**0**	**1**	**972.6**
180	14	14	54.8	66.9	**0**	**1**	**1,000**	**3.6**	**3.7**	**0**	**1**	**1,000**	10.3	10.5	**0**	**1**	**1,000**
240	18	1	43.1	60.1	21.51	267	**100**	10.1	13.6	16.31	18	**100**	**1.3**	**1.9**	**0**	**1**	**100**
240	18	2	8.38% (0)	14.17%	21.04	34,656	195.8	98	235.6	16.06	1,951	**196**	**1.9**	**2.4**	**0**	**1**	**196**
240	18	3	12.76% (0)	19.23%	21.51	12,929	283.2	856.6 (4)	7.38%	15.52	76,627	**286.2**	**4**	**4.5**	**0**	**1**	**286.2**
240	18	4	13.22% (0)	15.43%	20.3	5,288	367.8	7.77% (0)	11.38%	13.97	63,896	**372.8**	**5.1**	**6.9**	**0**	**1**	**372.8**
240	18	5	14.63% (0)	18.12%	21.16	2,672	440.2	8.07% (0)	10.45%	12.23	35,816	**457.2**	**8.8**	**10.7**	**0**	**1**	**457.6**
240	18	6	14.28% (0)	17.88%	19.27	1,675	515.6	8.59% (0)	10.63%	10.76	14,374	**535.4**	**14.5**	**16.4**	**0**	**1**	**536**
240	18	7	13.68% (0)	22.94%	18.57	561	578.8	7.52% (0)	9.26%	8.78	5,732	608	**19.6**	**20.8**	**0**	**1**	**612.4**
240	18	8	12.81% (0)	19.35%	17.01	350	636.6	5.91% (0)	8.86%	6.65	2,304	**673.6**	**29.1**	**31.9**	**0**	**1**	**683**
240	18	9	14.23% (0)	23.47%	18.61	236	668.8	5.06% (0)	7.1%	5.65	1,945	**722.8**	**32.8**	**34.9**	**0**	**1**	**740.6**
240	18	10	10.31% (0)	17.09%	12.9	164	736	4.16% (0)	5.5%	4.51	1,738	**766.2**	**38.2**	**45**	**0**	**1**	**783.4**
240	18	11	8.09% (0)	11.22%	9.92	139	786.4	2.87% (0)	3.82%	3.1	830	809	**45.2**	**49.6**	**0**	**1**	**819.6**
240	18	12	10% (0)	13.22%	12.18	81	797.8	2.7% (0)	4.66%	2.88	633	840	**47.8**	**58.6**	**0**	**1**	**852**
240	18	13	9.09% (0)	12.32%	11.01	87	830.4	1.26% (0)	1.91%	1.35	689	**879.6**	**55.1**	**61.3**	**0**	**1**	**880.2**
240	18	14	7.08% (0)	11.91%	8.64	70	869.8	1.31% (0)	1.75%	1.41	643	904	**61.9**	**67.2**	**0**	**1**	**906.8**
240	18	15	5.14% (0)	8.19%	6.36	67	907	1,509.7 (1)	1.24%	0.98	468	931	**66.5**	**72.1**	**0**	**1**	**932.2**
240	18	16	4.16% (0)	6.06%	5	162	934	1,449.9 (1)	0.8%	0.57	1,864	956	**66.5**	**75.8**	**0**	**1**	**957.2**
240	18	17	7.64% (0)	32.39%	11.06	115	912.8	367.2	665.2	0.2	10	**979.8**	**65.9**	**72.1**	**0**	**1**	**979.8**
240	18	18	164.2	202.2	**0**	**1**	**1,000**	**7.8**	**8.8**	**0**	**1**	**1,000**	20.5	21.2	**0**	**1**	**1,000**

be solved with linear programming. However, according to *nodes* many of the instances in F^{st} were solved in the root node by just adding preprocessing cuts. In general, among all tables, column *obj* shows how the objective function (i.e., passengers total satisfaction after the rescheduling) grows with the number of vehicles that are rescheduled. Block F^{sut} shows that formulation F^{st} is clearly outperformed by F^{sut}, which can be solved using liner programming.

From the results in Table 3 we first observe that F^{stk} provides worse running times and gaps than the previously analyzed F^{st} and F^{sut} formulations (see Table 2). As mentioned in Section 4.4, working with those formulations does not provide any advantage unless extra side constraints related to the features of different vehicles are included. Otherwise, we just enlarge the size of the formulation as well as the set of feasible symmetric solutions. On the contrary, although F^{sutk} is still solved as a linear program, the running times of the larger instances ($|T| = 240$) show an increase in the difficulty of solving these problems. Finally, our implementation of the model F^{sutk} given in Mesa, Ortega, and Pozo (2013) is clearly outperformed by any of the other formulations in terms of running time and gaps.

To summarize, formulation F^{sut} can be used to solve large instances in running times lower than 1 second, which fulfills the requirements for an efficient on-line rescheduling. In addition, we could cope with a bigger formulation F^{sutk} in reasonable times (1 minute in the worst case). This is an improvement with respect to the results of Mesa, Ortega, and Pozo (2013). As to formulations F^{stk} and F^{sutk}, even when the results provided are not competitive with F^{st} and F^{sut}, respectively, they use the index k, which might be very useful if other side constraints (such as capacities or different travel times for each vehicle) are added to the formulation. In this latter case, we refer to the heuristic approaches developed in Mesa, Ortega, and Pozo (2013).

5.2. Application to Real Data

We have tested the presented methodology in a real instance of the commuter train systems of Madrid. Demand data was obtained from a counting and a survey carried out in 2008 (see Mesa, Ortega, and Pozo 2009b, 2013). The sample was obtained in the work days of November and it represents an average work day. Figure 3 shows a section of Line C4 ($\langle 1\rangle$ = Parla—$\langle 2\rangle$ = Getafe Sector 3—$\langle 3\rangle$ = Getafe Centro—$\langle 4\rangle$ = Las Margaritas Universidad—$\langle 5\rangle$ = Villaverde Alto—$\langle 6\rangle$ = Villaverde Bajo—$\langle 7\rangle$ = Atocha) that we consider in our study. Table 4 shows departure times (h:m) at stations of all trains that complete the itinerary Parla–Atocha in the time period [6:00, 9:00], as well as the number of passengers boarding trains at each station ($b_{s,h:m}$).

Note that the time slots t corresponding to a departure time h:m can be obtained using $t = (h-6)\cdot 60 + m$.

Figure 3. (Color online) Line C4 (Parla–Atocha)

We have generated users arriving at station s at time t from passengers boarding trains at each station as follows: If t' and t'' are consecutive departure times at the same station, then $a_{st} = \lfloor b_{st''}/(t''-t')\rfloor + \lfloor t/t''\rfloor(b_{st''} - (t''-t')\lfloor b_{st''}/(t''-t')\rfloor)$ for $t \in (t', t'']$, that is, passengers arrive uniformly in $(t', t'']$ but we add to t'' those passengers whom we lost rounding down. Finally, the inconvenience cost function has been generated using (14) for values of $p \in \{1.5, 1.75, 2\}$.

Table 5 shows the objective values *obj* (also the percentage out of 32,206 passengers, *obj*%) when a reschedule is carried out with $\bar{\kappa}$ vehicles. First, we consider the *myopic* strategy that consists of canceling the $25-\bar{\kappa}$ timetables that serve the least number of users. Second, we compute the objective values obtained from formulation F^{sut} when the different values of $p = \{1.5, 1.75, 2\}$ are considered for the inconvenience cost function $\tilde{\varphi}$. Note that in this case every instance has been optimally solved in less than one second. From the results depicted in Table 5 we mainly conclude that the best values are obtained for $p = 2$ whereas the values closer to the *myopic* solution (i.e., the worst values) are obtained for $p = 1.5$. This means that the

Table 4. Timetables and Boarding in Line C4

	$s=1$		$s=2$		$s=3$		$s=4$		$s=5$		$s=6$		$s=7$	
k	$h{:}m$	$b_{s,h:m}$	$h{:}m$	$b_{s,h:m}$	$h{:}m$	$b_{s,h:m}$	$h{:}m$	$b_{s,h:m}$	$h{:}m$	$b_{s,h:m}$	$h{:}m$	$b_{s,h:m}$	$h{:}m$	$b_{s,h:m}$
1	6:04	335	6:10	1	6:13	44	6:15	7	6:18	44	6:21	46	6:31	147
2	6:12	177	6:18	5	6:21	113	6:24	48	6:26	124	6:29	81	6:38	302
3	6:16	307	6:22	1	6:25	35	6:28	29	6:30	64	6:34	58	6:42	123
4	6:22	55	6:28	8	6:31	138	6:33	54	6:36	163	6:39	86	6:48	234
5	6:28	429	6:34	10	6:36	145	6:39	62	6:42	173	6:44	119	6:54	349
6	6:34	511	6:40	4	6:42	102	6:44	26	6:46	153	6:50	115	7:00	571
7	6:40	484	6:46	12	6:48	151	6:50	54	6:52	119	6:57	107	7:07	129
8	6:46	491	6:52	10	6:54	166	6:56	70	7:00	157	7:04	160	7:12	184
9	6:54	414	7:00	24	7:03	254	7:05	119	7:08	158	7:12	185	7:20	490
10	7:01	476	7:07	17	7:10	195	7:12	88	7:15	106	7:18	145	7:26	514
11	7:09	421	7:15	33	7:18	260	7:20	111	7:24	146	7:27	209	7:35	491
12	7:16	550	7:22	38	7:25	218	7:27	158	7:30	123	7:34	298	7:43	574
13	7:22	414	7:28	36	7:31	247	7:34	119	7:38	136	7:41	243	7:48	451
14	7:28	421	7:34	26	7:37	127	7:41	104	7:44	97	7:47	154	7:54	276
15	7:34	386	7:40	31	7:42	145	7:45	113	7:48	103	7:53	144	8:00	424
16	7:40	384	7:46	47	7:48	171	7:50	108	7:54	99	7:58	180	8:06	284
17	7:46	323	7:52	31	7:54	202	7:57	134	8:00	128	8:04	231	8:12	647
18	7:52	408	7:58	19	8:01	190	8:02	77	8:05	84	8:09	192	8:18	446
19	7:58	441	8:03	49	8:06	210	8:09	91	8:13	119	8:15	223	8:24	335
20	8:04	165	8:10	47	8:13	229	8:15	110	8:19	126	8:23	259	8:30	338
21	8:11	347	8:15	44	8:18	225	8:21	134	8:25	98	8:28	165	8:36	302
22	8:16	336	8:22	38	8:25	294	8:28	112	8:30	79	8:33	158	8:42	271
23	8:22	317	8:28	33	8:31	230	8:34	119	8:36	119	8:40	156	8:48	410
24	8:28	335	8:34	36	8:37	119	8:39	91	8:41	61	8:45	144	8:54	364
25	8:34	265	8:40	13	8:43	117	8:45	66	8:47	43	8:52	153	9:00	381

Table 5. Objective Values When a Reschedule Is Carried Out with $\bar{\kappa}$ Vehicles

	myopic		$F^{sut} \mid p=1.5$		$F^{sut} \mid p=1.75$		$F^{sut} \mid p=2$	
$\bar{\kappa}$	*obj*	*obj%*	*obj*	*obj%*	*obj*	*obj%*	*obj*	*obj%*
1	1,959	6.08	2,133	6.62	2,225	6.91	2,304	7.15
2	3,655	11.35	4,002	12.43	4,166	12.94	4,310	13.38
3	5,326	16.54	5,858	18.19	6,103	18.95	6,314	19.61
4	6,972	21.65	7,615	23.64	7,902	24.54	8,156	25.32
5	8,616	26.75	9,285	28.83	9,646	29.95	9,961	30.93
6	10,157	31.54	10,909	33.87	11,325	35.16	11,693	36.31
7	11,639	36.14	12,506	38.83	12,984	40.32	13,416	41.66
8	13,107	40.7	14,084	43.73	14,557	45.20	15,042	46.71
9	14,523	45.09	15,580	48.38	16,083	49.94	16,567	51.44
10	15,907	49.39	17,052	52.95	17,561	54.53	18,067	56.10
11	17,253	53.57	18,418	57.19	18,897	58.68	19,430	60.33
12	18,568	57.65	19,733	61.27	20,191	62.69	20,668	64.17
13	19,856	61.65	21,011	65.24	21,463	66.64	21,892	67.97
14	21,143	65.65	22,203	68.94	22,599	70.17	22,987	71.37
15	22,417	69.61	23,320	72.41	23,664	73.48	24,019	74.58
16	23,690	73.56	24,435	75.87	24,729	76.78	25,041	77.75
17	24,928	77.4	25,535	79.29	25,780	80.05	26,045	80.87
18	26,133	81.14	26,579	82.53	26,783	83.16	27,037	83.95
19	27,283	84.71	27,578	85.63	27,769	86.22	27,970	86.85
20	28,339	87.99	28,554	88.66	28,715	89.16	28,869	89.64
21	29,377	91.22	29,480	91.54	29,614	91.95	29,739	92.34
22	30,227	93.86	30,393	94.37	30,474	94.62	30,556	94.88
23	30,965	96.15	31,294	97.17	31,330	97.28	31,366	97.39
24	31,589	98.08	31,796	98.73	31,817	98.79	31,840	98.86
25	32,206	100	32,206	100	32,206	100	32,206	100

Figure 4. (Color online) Timetables and *myopic* Rescheduling (Bold) for 15 Vehicles

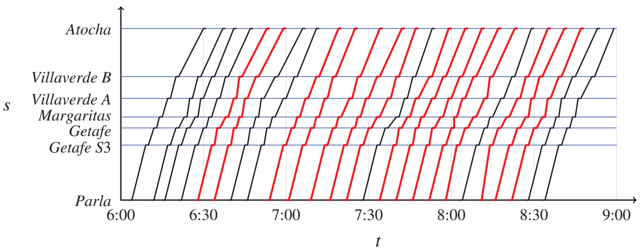
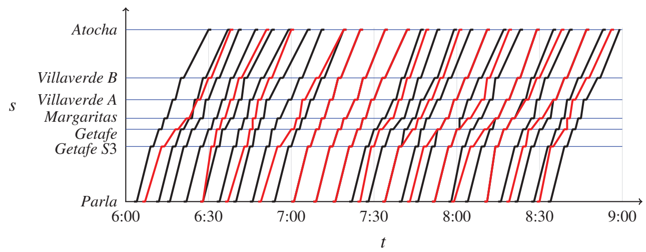


Figure 5. (Color online) Timetables and F^{sut} Solution (Bold) for 15 Vehicles



inconvenience grows faster as long as p decreases, and consequently, the smaller the value of p the closer the objective values of F^{sut} are to those of the *myopic* strategy.

Figure 4 shows the 25 initial timetables provided by the company, and by bold lines, the timetables of the *myopic* solution when a reschedule of 15 vehicles is carried out. On the other hand, Figure 5 shows the 25 initial timetables provided by the company, and by bold lines, those timetables obtained for the optimal rescheduling of 15 trains according to the developed model. If the myopic selection of the 15 most efficient timetables were decided, the objective value (i.e., total convenience) would be 22,417. An improvement of 1,602 units could be reached if the rescheduling is performed by applying the model to determine the 25 optimal line runs.

6. Conclusions

In this paper we have presented a modeling approach for solving the rescheduling problem in a transit line that has suffered a fleet size reduction. We have described a demand pattern to reflect the passengers' behavior when some vehicle services are delayed or canceled. This inconvenience function has been used to derive a rescheduling framework coming from a timetabling formulation. We have shown that the problem can be rapidly solved using a formulation whose coefficient matrix we prove is totally unimodular. We have tested the different formulations over a testbed of random instances: The results show that on-line rescheduling can be done efficiently using the proposed models and that previous approaches in the literature are outperformed.

This paper does not consider some extensions that could be included in future research. First, formulations that use the index k (for the different vehicles) might be very useful if other side constraints are added. There are some kinds of constraints that could be added without losing the integrality property, for example using the waiting time of vehicle k at station s (λ_{sk}) or the travel time of vehicle k from station s to station $s+1$ (μ_{sk}). Second, additional constraints could be added related to bounds on the variables, e.g., time windows on departing/arriving vehicles. As to vehicle capacities, including capacity constraints would destroy the linear properties of the model. In our formulations, we can only penalize delays/cancelations harder in those stations with an intense level of demand. This would be done by just properly calibrating the inconvenience function $\varphi_{st't}$. Eventually, the operator may decide to add additional wagons in overcrowded trains (once the rescheduling is computed). Issues concerning capacities in vehicle rescheduling can be revised in Kroon, Maróti, and Nielsen (2015). Finally, as in Mesa, Ortega, and Pozo (2013), this paper does not consider the way back of vehicles in the transit line. For such type of models in the timetabling and scheduling problem, we refer the reader to Mesa et al. (2014b).

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Appendix A. Proofs

Property 1. *Let Ω^{st} be the lattice points defined by constraints* (6b)–(6f) *and $\vec{\Omega}^{sut;x}$ the projection defined by constraints* (10b)–(10i) *over the x variables. Then $\Omega^{st} = \vec{\Omega}^{sut;x}$.*

Proof. We prove first that every feasible solution to Ω^{st} is feasible in $\vec{\Omega}^{sut;x}$. Given $x \in \Omega^{st}$ we prove that there exists z, given by (9), thus verifying $(x,z) \in \vec{\Omega}^{sut}$, that is, (x,z) has to verify (10c) and (10d). From (10c) and (9) we must prove that

$$x_{st} = \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} z_{s+1,t+\mu_s,t'} = \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} x_{st}x_{s+1,t'} = x_{st}\sum_{t':(t+\mu_s,t')\in T^2_{s+1}} x_{s+1,t'},$$

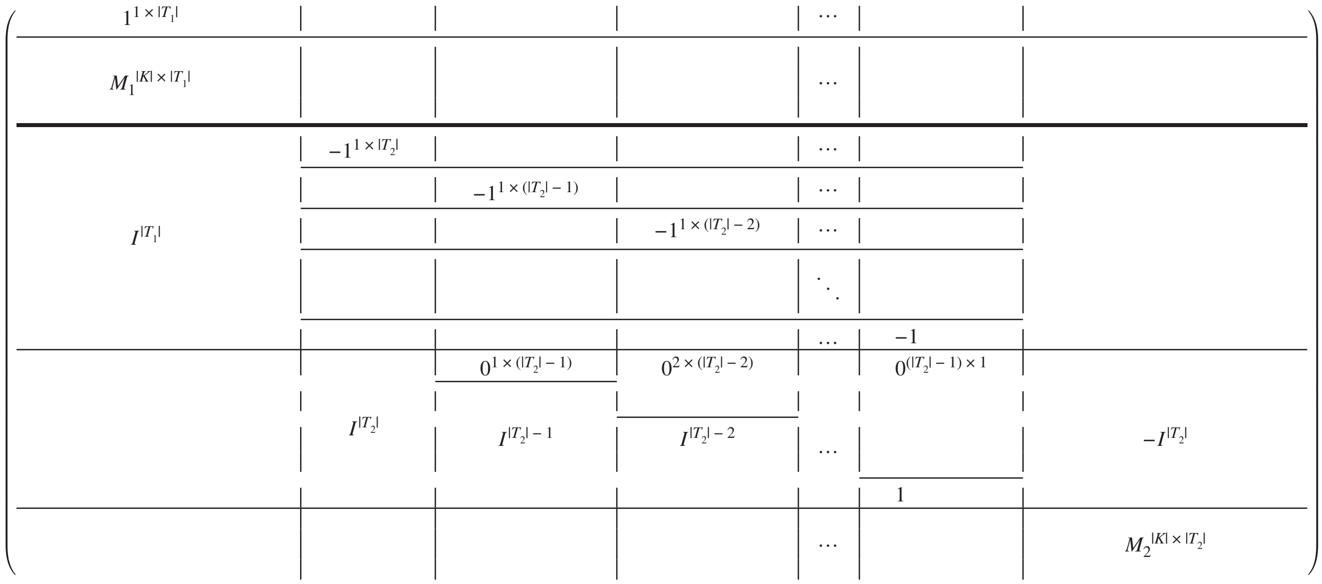
which is true when $x_{st}=0$. If $x_{st}=1$, then we have $1 = \sum_{t':(t+\mu_s,t')\in T^2_{s+1}} x_{s+1,t'}$ and therefore (10c) holds.

From (10d) and (9)

$$x_{st} = \sum_{u:(u,t)\in T^2_s} z_{sut} = \sum_{u:(u,t)\in T^2_s} x_{s-1,u-\mu_{s-1}}x_{st} = x_{st}\sum_{u:(u,t)\in T^2_s} x_{s-1,u-\mu_s-1},$$

which is true when $x_{st}=0$. If $x_{st}=1$, then we have $1 = \sum_{u:(u,t)\in T^2_s} x_{s-1,u-\mu_s-1}$ and therefore (10d) holds.

Figure A.1. Structure of $\vec{A}^{sut}$ Matrix When $s \in \{1,2\}$, $(u,t) \in T_s^2$, and $\mu_s = 0$



Therefore $\Omega^{st} \subseteq \vec{\Omega}^{sut;x}$.

Conversely, we prove that every feasible solution $(x,z) \in \vec{\Omega}^{sut;x}$ is feasible in Ω^{st} if x it verifies (6c).

From (10c) and (9)

$$\begin{aligned} x_{st} &= \sum_{t':(t+\mu_s,t')\in T_{s+1}^2} z_{s+1,t+\mu_s,t'} = \sum_{t':(t+\mu_s,t')\in T_{s+1}^2} x_{st} x_{s+1,t'} \\ &= x_{st} \sum_{t':(t+\mu_s,t')\in T_{s+1}^2} x_{s+1,t'} \le \sum_{t':(t+\mu_s,t')\in T_{s+1}^2} x_{s+1,t'}. \end{aligned}$$

This inequality concludes the proof. □

Property 2. *Let Ω_{LR}^{st} be the polyhedron defined by constraints (6b)–(6e) and $\vec{\Omega}_{LR}^{sut;x}$ the projection defined by constraints (10b)–(10g) over the x variables. Then $\Omega_{LR}^{st} \supsetneq \vec{\Omega}_{LR}^{sut;x}$.*

Proof. Every fractional solution $(x,z) \in \vec{\Omega}^{sut}$ verifies that $x \in \Omega^{st}$ since (6c) is implied by (10c). Therefore $\Omega_{LR}^{st} \supset \vec{\Omega}_{LR}^{sut;x}$.

On the contrary, there are fractional solutions $x \in \Omega^{st}$ that do not belong to $\vec{\Omega}^{sut}$ (as shown in Section 5). □

Property 3. *The RFRP can be solved with linear programming by means of $\vec{F}^{sut}$.*

Proof. Denoting by $\vec{A}^{sut}$ the matrix of coefficients coming from problem $\vec{F}^{sut}$, we prove that $\vec{A}^{sut}$ is TU.

Note that $(\vec{A}^{sut})_{s\in\{1,2\},(u,t)\in T_s^2}$ when $\mu_s = 0$ has the form indicated in Figure A.1, where empty boxes represent zeros, $1^{m\times n}$ stands for a $m \times n$ matrix of all ones (analogously for 0 and −1), I^n is the identity matrix in dimension n, and $M_s^{m\times n}$ is a matrix of consecutive ones (interval matrix) by the rows and columns so that it has at most two ones per column.

To prove that $\vec{A}^{sut}$ is TU we give, without loss of generality, the argument for $(\vec{A}^{sut})_{s\in\{1,2\},(u,t)\in T_s^2}$ since it remains the same for a general number of vehicles $s > 2$. It is known that a matrix $A = (a_{ij})$ is TU if and only if for every $J \subseteq N = (1,\dots,n)$ there exists a partition J_1, J_2 of J such that

$$\left| \sum_{j\in J_1} a_{ij} - \sum_{j\in J_2} a_{ij} \right| \le 1, \quad \forall\, i = 1,\dots,m. \tag{A.1}$$

The proof is constructive. We choose an arbitrary (sorted) subset of columns of $\vec{A}^{sut}$, $J = (j_1,\dots,j_l)$ and we construct J_1 and J_2 as follows: We start with $J_1' = j_1$ and $J_2' = \emptyset$. Iteratively, we try if (A.1) is fulfilled with $J_1 = J_1' \cup j_k$ and $J_2 = J_2'$. If so, we redefine $J_1' := J_1' \cup j_i$ and otherwise, we redefine $J_2' := J_2' \cup j_k$. Assume that (A.1) holds for given intermediate sets J_1' and J_2' after $k-1$ steps of the above process. Next, if the addition of j_k to current J_1' violates condition (A.1), the structure of the matrix $\vec{A}^{sut}$ ensures that putting j_k to J_2' keeps the value of the difference $-1 \le \sum_{j\in J_1'} a_{ij} - \sum_{j\in J_2'\cup\{j_k\}} a_{ij} \le 1$, $\forall\, i = 1,\dots,m$. Repeating this process for all $k = 1,\dots,l$ we obtain a decomposition of J into $J_1 = J_1'$ and $J_2 = J_2'$ that satisfies (A.1) which proves the claim. □

Property 4. *Let Ω^{sut} be the lattice points defined by constraints (11b)–(11f) and $\vec{\Omega}^{sut;z}$ the projection of (10b)–(10i) over the z variables. Then $\Omega^{sut} = \vec{\Omega}^{sut;z}$.*

Proof. We prove first that every feasible solution $(x,z) \in \vec{\Omega}^{sut;z}$ is feasible in Ω^{sut}. First, equating constraints (10c) and (10d) results in (11c). Second, constraints (10b) and (10d) imply (11b) since

$$\bar{\kappa} = \sum_{t\in T_1} x_{1t} = \sum_{t\in T_2} x_{2t} = \sum_{t\in T_2} \sum_{u:(u,t)\in T_2^2} z_{2ut} = \sum_{(u,t)\in T_2^2} z_{2ut} = \bar{\kappa}.$$

Finally, the same argument can be applied to show that (10d) and (10e) imply (11b). Indeed

$$\begin{aligned} \sum_{(u,t)\in T_s^2:\,\theta_{s,k-1}^- < t < \theta_{s,k+1}^-} z_{sut} &= \sum_{t\in T_s:\,\theta_{s,k-1}^- < t < \theta_{s,k+1}^-} \sum_{u:(u,t)\in T_s^2} z_{sut} \\ &= \sum_{t\in T_s:\,\theta_{s,k-1}^- < t < \theta_{s,k+1}^-} x_{st} \le 1. \quad \square \end{aligned}$$

Property 5. *Let Ω_{LR}^{sut} be the polyhedron defined by constraints (11b)–(11e) and $\vec{\Omega}_{LR}^{sut;z}$ the projection of (10b)–(10g) over the z variables. Then $\Omega_{LR}^{sut} = \vec{\Omega}_{LR}^{sut;z}$.*

Proof. The proof follows directly from the proof for Property 4 since it does not use the integer character of the solutions, and thus is also valid for fractional solutions. □

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